Tutorial on Cognitive Logics

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Outline I

Talk Overview

Section 1

How human reasoning deviates from classical logic

Basics of propositional logic

$\mathcal{L} = \mathcal{L}(\Sigma)$	propositional language $\mathcal{L}$ over a set of atoms Σ
$\neg, \wedge, \vee$	junctors for negation, conjunction, disjunction
$A \Rightarrow B$	$\equiv \neg A \vee B$ material implication
Ω	set of interpretations/models/possible worlds over Σ
$\omega \models A$	ω is a model of $A (\in \mathcal{L})$
$\mathit{Mod}(A)$	set of models of A
$A \models B$	iff $\mathit{Mod}(A) \subseteq \mathit{Mod}(B)$ classical deduction
$\mathit{Cn}(A)$	$= \{B \in \mathcal{L} \mid A \models B\}$ classical consequence operator

Classical inference rules

Modus ponens $\dfrac{A \Rightarrow B,\ A}{B}$

Modus tollens $\dfrac{A \Rightarrow B,\ \neg B}{\neg A}$

Monotony $\dfrac{A \Rightarrow B}{A \wedge C \Rightarrow B}$

Transitivity $\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$

Classical properties/axioms: Contraposition

From $A \models B$ conclude $\neg B \models \neg A$

$Penguin \models Bird$	*Penguins are birds.*
$\neg Bird \models \neg Penguin$	*Non-birds are non-penguins.* :)

$Human\_being \mid\!\sim \neg Millionaire$

Humans usually are not millionnaires.

$Millionaire \mid\!\sim \neg Human\_being$

Millionnaires usually are not human. :(

Classical properties/axioms: Transitivity

From $A \models B$ and $B \models C$ conclude $A \models C$

$Penguin \models Bird$	*Penguins are birds.*
$Bird \models Animal$	*Birds are animals.*
$Penguin \models Animal$	*Penguins are animals.* :)

$Penguin \mid\!\sim Bird$	*Penguins are birds.*
$Bird \mid\!\sim Fly$	*Birds can fly.*
$Penguin \mid\!\sim Fly$	*Penguins can fly.* :(

Classical properties/axioms: Monotony

From $A \models C$ conclude $A \wedge B \models C$

$\dfrac{Penguin \models Bird}{Penguin \wedge Black \models Bird}$	*Penguins are birds.* / *Black penguins are birds.*	:)
$\dfrac{Bird \mid\!\sim Fly}{Bird \wedge Penguin \mid\!\sim Fly}$	*Birds can fly.* / *Pegnguin-birds can fly.*	:(

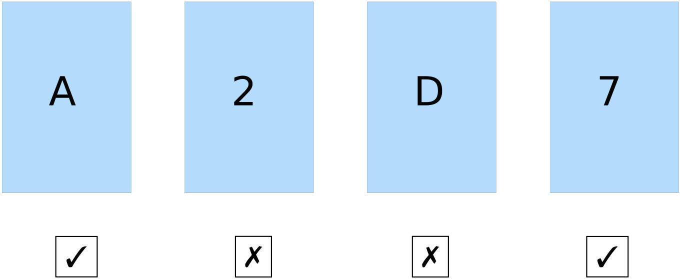
From the common sense perspective, classical logic is inadequate, now let's have a look on the cognitive perspective . . .

Talk Overview

Section 2

Cognitive perspective

Observation 1: The Wason Selection Task [21]



- Given:
  - Four cards with a letter on one and a number on the other side
  - A rule: *If there is a vowel on one side there is an even number on the other side*
- Decide:
  - Exactly which cards needs to be turned in order to check that the rule holds?

Observation 1': The Deontic Case [3]

Again 4 cards; on one side person's age/backside drink.

If a person is drinking beer, then the person must be over 19 years of age.

Which cards must be turned to prove that the conditional holds?

	beer	coke	22yrs	16yrs
Experimental Results	95%	2.5%	2.5%	80%

- Isomorphic to the previous problem. But, most get it right!
- Observations:
  - Humans can reason classically logically, but not always
  - Even for isomorpic problems human reasoning is not equivalent

Meta-analysis of WST [17]

- Pubmed, Science Direct, or Google Scholar search with keywords: (conditional reasoning) or (selection task) or (Wason card)
- Inclusion of studies that report
  - Rules: if p, then q; every p
  - Individual selection patterns (No aggregation!)
  - At least the four canonical selections: p, pq, $p\bar{q}$, $pq\bar{q}$ per Ss
- Inclusion of 228 experiments with N $=$ 18,000 Ss:
  - Abstract: 104 exp; Everyday: 44 exp; Deontic: 80 exp
- Aggregated results for the canonical selections in %

	p	pq	$pq\bar{q}$	$p\bar{q}$
Abstract	36	39	5	19
Everyday	23	37	11	29
Deontic	13	19	4	64

- Data can be found here: [17] and `https://www.cc.uni-freiburg.de/data/`

Observation 2a: Belief Bias [5]

All frenchmen drink wine
Some wine drinkers are gourmets
———————————————
Some frenchmen are gourmets

Although the argument is widely accepted, it is not valid!

All frenchmen drink wine
Some wine drinkers are italians
———————————————
Some frenchmen are italians

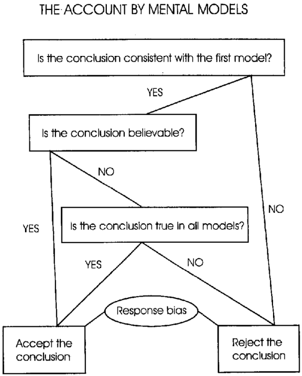
- Belief (in conclusion) Bias Effect!

Observation 2: Belief Bias – Meta-Analysis

Conclusion	Syllogism	
	Believable	Unbelievable
Valid	No cigarettes are inexpensive.	No addictive things are inexpensive.
	Some addictive things are inexpensive.	Some cigarettes are inexpensive.
	Therefore, some addictive things are not cigarettes.	Therefore, some cigarettes are not addictive.
	P("valid") = .92	P("valid") = .46
Invalid	No addictive things are inexpensive.	No cigarettes are inexpensive.
	Some cigarettes are inexpensive.	Some addictive things are inexpensive.
	Therefore, some addictive things are not cigarettes.	Therefore, some cigarettes are not addictive.
	P("valid") = .92	P("valid") = .08

Example and numbers taken from [19].

Observation 2: Belief Bias – Meta-Analysis [19]



- Can be explained by
  - Background knowledge
  - Erroneously reasoning about consistency instead of deductive reasoning
  - Humans focusing on the conclusion instead on the reasoning process

Picture taken from [12].

- Data can be found here: `https://osf.io/8dfyv/`

Observation 2b: Knowledge frame [20]

Linda is 31 Jahre old, single, outspoken and very intelligent. As a student she concerned herself thoroughly with subjects of discrimination and social justice and participated in protest against nuclear energy.

Rank the following statements by their probabilities.

- Linda works as a bank teller.
- Linda works as a bank teller and is an active feminist.

- Result: More than 80% judge Linda works as a bank teller and is an active feminist to be more likely than Linda works as a bank teller.
- BUT: $p(a \wedge b) \leqslant p(a)$ or $p(b)$
- Hence, most answer falsely from the perspective of probability!
- Instead humans use the so called representativity heuristic.

Observation 3: Nonmonotonicity

- If Lisa has an essay to write, Lisa will study late in the library
- If the library is open, Lisa will study late in the library
- Lisa an essay to write

  - 🔴 Lisa will study late in the library
  - 🟡 Nothing follows
  - 🟢 Can't say or I have another solution

The Suppression Task [2]

- *If she has an essay to write, she will study late in the library.*
- *If the library is open, she will study late in the library.*
- *She has an essay to write.*

95% of all subjects conclude (modus ponens): Only 60% of all subjects conclude:

- She will study late in the library.

A logic is called non-monotonic if the set of (logical) conclusions from a knowledge base is not necessarily preserved when new information is added to the knowledge base.

- Everyday reasoning is often non-monotonic [18, 9]

Suppression Task

Facts	Conditional	Alternative Argument	Additional Argument
	If she has an essay to finish, then she will stay late in the library	If she has a textbook to read, then she will stay late in the library	If the library stays open, then she will stay late in the library
She has an essay to finish	She will study late in the library (96% L)	She will study late in the library (96% L)	She will study late in the library (38% L)
She does not have an essay to finish	She will not study late in the library (46% $\neg L$)	She will not study late in the library (4% $\neg L$)	She will not study late in the library (63% $\neg L$)

Additional arguments lead to the suppression of previously drawn conclusions.
Alternative Arguments lead to the suppression of previously drawn conclusions.

Suppression Task: Classical Logic

If she has an essay to finish	then she will stay late in the library	$l \leftarrow e$
If she has a textbook to read	then she will stay late in the library	$l \leftarrow t$
If the library stays open	then she will stay late in the library	$l \leftarrow o$

Clauses	Facts	Classical Logic	Exp. Findings	
$l \leftarrow e$	e	$\models\ l$	96% L	Modus Ponens
$l \leftarrow e \quad l \leftarrow t$	e	$\models\ l$	96% L	Modus Ponens
$l \leftarrow e \quad l \leftarrow o$	e	$\models\ l$	38% L	Modus Ponens
$l \leftarrow e$	$\neg e$	$\not\models \neg l$	46% $\neg L$	Denial of the Antecedent
$l \leftarrow e \quad l \leftarrow t$	$\neg e$	$\not\models \neg l$	4% $\neg L$	Denial of the Antecedent
$l \leftarrow e \quad l \leftarrow o$	$\neg e$	$\not\models \neg l$	63% $\neg L$	Denial of the Antecedent

Classical logic does not adequately represent the suppression task.

RQ 1: Are card selections (inference rules) cognitively dependent or independent?

- Some studies reported only percentage of selections for the 4 cards
- Caveat: Requires that selection of a card is independent from others
  - Some analysis report independence [4]
  - But other analysis correlations between pairs of selections [15, 13]
- Who is right . . . and how can we test this?
- Idea: Combine Shanon's measure of entropy with simulations of thousands of experiments
  - Entropy is a measure of unpredictability of the state (0 = certain)

$$H = -\sum p_i \log_2 p_i$$

  - If H(card selections in experiment) reliably smaller as H(card selections in simulations) then card selection in experiment are dependent

Entropy of the 228 experiments and 10K simulations

Data: Exp. with no. of Ss and frequencies of the four selections
Result: Proportion of experiments with lower/higher entropy
foreach *experiment* **do**
  `Compute(`*N, percentage, probs of selection for each of the 4 cards*`)`
  `Compute(`*Shanon's entropy H for the experiment*`)`
  `Simulate(`*10K experiments based on the probs of selecting each card*`)`
end

Three sorts of selection task	Mean entropy of experiments	Mean entropy of 10K simulations	Wilcoxon's W and p-value
Abstract	1.32	1.42	$W = 469$, $p < .001$
Everyday	1.51	1.66	$W = 28$, $p < .001$
Deontic	1.06	1.21	$W = 68$, $p < .001$

- Independence of card selections by Ss can be rejected!

Independence assumption of theories

- Independence of card selections by Ss can be rejected!
- Theories assuming independence are built on false assumptions!
- (Eliminates 13 existing cognitive theories)

RQ 2: Insufficiency of two-valued interpretation

Rule Number	Name	Premises	Conclusion	Logically correct?
1	MP	$p \rightarrow q, p$	q	Yes
2	DA	$p \rightarrow q, \overline{p}$	$\overline{q}$	No
3	AC	$p \rightarrow q, q$	p	No
4	MT	$p \rightarrow q, \overline{q}$	$\overline{p}$	Yes

Figure: Inference rules and abbreviations: MP: Modus Ponens, DA: Denial of Antecedent, AC: Affirmation of Consequence, MT: Modus Tollens.

Individual answer pattern matters

- Often focused on aggregated responses for each card
- Instead each individual answer pattern is more sensible
- Overall there can be 2^4 distinct answer patterns
- Conducted meta-analysis (43 Exp) for individual patterns

An interesting pattern

- How often is $\mathrm{MP} + \mathrm{MT} + \mathrm{AC}$ chosen in each study?
  $\rightarrow$ And replicable by any two-valued valuation?

Meta-Analysis of Wason Selection Task [16]

Six patterns in the meta-analysis of 46 articles. Ss = number of participants. All other values are percentages chosen by the participants

Publication	Ss	MP p	MP + MT $p, \overline{q}$	MP + AC p, q	MP + MT + AC $p, q, \overline{q}$	MP + AC + DA $p, q, \overline{p}$	All $p, q, \overline{p}, \overline{q}$	Oth
Social								
[3]	32	44	9	31	9	0	0	7
[8]	50	6	82	2	6	0	0	4
[22]	40	0	65	3	25	0	0	8
[22]	40	0	45	18	8	0	0	1
[7]	60	27	17	23	10	0	0	2
[6]	25	16	16	36	12	0	0	2
Total	247	15	42	17	11	0	0	1
Abstract								
[10]	128	33	4	46	7	0	0	1
[14]	12	33	33	25	8	0	0	0
[23]	320	19	36	13	6	2	8	1
[8]	50	28	0	52	6	0	0	1
[1]	16	13	25	25	19	0	6	0
[11]	89	13	19	24	9	2	13	1
[18]	n/a	35	5	45	7	n/a	n/a	8
Total	615	18	13	40	7	1	2	1

Two-valued valuations [16]

- Each statement is mapped to 1 (true) or 0 (false)
- Idea: Search through the space of all possible valuations to explain the 2^4 reasoning patterns, especially if chosen
- We use valuations of the form $p \rightarrow_\chi q$ where index χ denotes the pattern that follows from the valuation of $\rightarrow_\chi$
- Patterns indicate a different inference process, i.e., the 16 patterns are unique
  - E.g., $\mathrm{MP} + \mathrm{AC}$ does not correspond to MP, since in the latter AC is not chosen

Example [16]

Table: n/a means that this is not relevant when evaluating the implication

p	q	$p \to q$	$p \to_{MP} q$	$p \to_{AC} q$
$\bot$	$\bot$	$\top$	n/a	n/a
$\bot$	$\top$	$\top$	n/a	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	n/a
$\top$	$\top$	$\top$	$\top$	$\top$

Example [16]

Table: n/a means that this is not relevant when evaluating the implication

p	q	$p \rightarrow q$	$p \rightarrow_{MP} q$	$p \rightarrow_{AC} q$
$\perp$	$\perp$	$\top$	n/a	n/a
$\perp$	$\top$	$\top$	n/a	$\perp$
$\top$	$\perp$	$\perp$	$\perp$	n/a
$\top$	$\top$	$\top$	$\top$	$\top$

Valuations

$$v(p \rightarrow_{\mathsf{MP}} q) = \begin{cases} 0 & \text{if } v(p) = 1 \text{ and } v(q) = 0 \\ 1 & \text{if } v(p) = v(q) = 1 \\ 0/1 & \text{otherwise} \end{cases}$$

$$v(p \rightarrow_{\mathsf{MT}} q) = \begin{cases} 0 & \text{if } v(p) = 1 \text{ and } v(q) = 0 \\ 1 & \text{if } v(p) = v(q) = 0 \\ 0/1 & \text{otherwise} \end{cases}$$

$$v(p \rightarrow_{\mathsf{AC}} q) = \begin{cases} 0 & \text{if } v(q) = 1 \text{ and } v(p) = 0 \\ 1 & \text{if } v(p) = v(q) = 1 \\ 0/1 & \text{otherwise} \end{cases}$$

$$v(p \rightarrow_{\mathsf{DA}} q) = \begin{cases} 0 & \text{if } v(q) = 0 \text{ and } v(p) = 1 \\ 1 & \text{if } v(p) = v(q) = 0 \\ 0/1 & \text{otherwise} \end{cases}$$

Example [16]

Table: The pattern MP+MT+AC has the same evaluation as the "all four" pattern, which means it also satisfies DA

p	q	MP	MT	AC	MP+MT+AC	All four
$\bot$	$\bot$	$\top/\bot$	$\top$	$\top/\bot$	$\top$	$\top$
$\bot$	$\top$	$\top/\bot$	$\top/\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top/\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top/\bot$	$\top$	$\top$	$\top$

Example [16]

Table: The pattern $\mathrm{MP} + \mathrm{MT} + \mathrm{AC}$ has the same evaluation as the "all four" pattern, which means it also satisfies DA

p	q	MP	MT	AC	MP+MT+AC	All four
$\bot$	$\bot$	$\top/\bot$	$\top$	$\top/\bot$	$\top$	$\top$
$\bot$	$\top$	$\top/\bot$	$\top/\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top/\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top/\bot$	$\top$	$\top$	$\top$

Example

Table: The pattern MP+MT+AC has the same evaluation as the "all four" pattern, which means it also satisfies DA

p	q	MP	MT	AC	MP+MT+AC	All four
$\bot$	$\bot$	$\top/\bot$	$\top$	$\top/\bot$	$\top$	$\top$
$\bot$	$\top$	$\top/\bot$	$\top/\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top/\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top/\bot$	$\top$	$\top$	$\top$

Consequences [16]

Lemma

The relation between the inference rules defined in the table are:

- $\rightarrow_{MT+AC}$ *holds if and only if* $\rightarrow_{MP+DA}$ *holds*
- $\rightarrow_{MT+AC}$ *implies* $\rightarrow_{DA}$
- $\rightarrow_{MP+DA}$ *implies* $\rightarrow_{AC}$
- $\rightarrow_{MT+AC+DA}$ *holds if and only if* $\rightarrow_{MP+DA+AC}$ *holds*
- $\rightarrow_{MP+MT+AC}$ *holds if and only if* $\rightarrow_{MP+MT+DA}$ *holds*

Corollary

There is no two-valued valuation for the patterns $MT + AC$, $MP + DA$, $MT + AC + DA$, $MP + DA + AC$, $MP + MT + AC$, $MP + MT + DA$

Three-valued logics [16]

- Core idea: Assign three values (0, u, 1) to p and q.
- As we model the Wason Selection Task the valuations of the Boolean functions are mapped to the set $\{0,1\}$ with 1 "turn" and 0 "not turn".
- Alloews for more freedom of interpretation, and allows us to find a uniquely represent all the patterns that were missing under binary logics.
- The lemma does not hold for ternary logics, since there are at least two unique truth tables that satisfy each of the 16 reasoning patterns.

Truth table for three-valued logics

p	q	$\rightarrow_{MP}$	$\rightarrow_{MT}$	$\rightarrow_{MP+MT}$	$\rightarrow_{AC}$	$\rightarrow_{DA}$
0	0	0/1	1	1	0/1	1
0	1	0/1	0/1	0/1	0	0
1	0	0	0	0	0/1	0/1
1	1	1	0/1	1	1	0/1
0	u	0/1	0/1	0/1	0/1	0
u	0	0/1	0	0	0/1	0/1
u	u	0/1	0/1	0/1	0/1	0/1
u	1	0/1	0/1	0/1	0	0/1
1	u	0	0/1	0	0/1	0/1

Figure: The light grey values are the ones chosen for $\rightarrow_{MP+MT+AC}$, while the dark grey values are the ones chosen for $\rightarrow_{MP+MT+DA}$.

Formal inference methods

Do formal nonmonotonic inference approaches show this behavior?

- Change of perspective:
  - From: Use formal inference systems as a norm for correct human behavior ($\rightarrow$ deviations of human reasoning)
  - To: Use human "commonsense" reasoning to evaluate formal inference methods ($\rightarrow$ cognitive-adequacy of formalisms)
- Already known: Logic Programming with weak completion semantics shows suppression effect. [Stenning and Lambalgen, 2008]
- However there are many other approaches, e.g.,
  - System P
  - System Z
  - Reiter Default Logic
  - c-Representations
  - c-Representations + Revision

$\Rightarrow$ See Section 3

Intermediate summary

- Instead of analyzing aggregated values, single responses provide the "real" inference process.
- Human reasoners generate patterns that can not be reproduced by classical logic approaches.
- Some answer patterns have implications for other answer patterns.
- Three-valued logics can explain the answer results.

Classical consequences

Consequence operator of classical logic:

$$\begin{aligned} \mathcal{C}n &: 2^{\mathcal{L}} \to 2^{\mathcal{L}} \\ \mathcal{C}n(\mathcal{F}) &:= \{G \in \mathcal{L} \mid \mathcal{F} \models G\} \end{aligned}$$

$\mathcal{C}n$ is monotone, i.e., from $\mathcal{F} \subseteq \mathcal{G}$ we conclude $\mathcal{C}n(\mathcal{F}) \subseteq \mathcal{C}n(\mathcal{G})$

A set of formulas $\mathcal{F}$ is *closed (deductively)* iff $\mathcal{C}n(\mathcal{F}) = \mathcal{F}$.

Deduction theorem relates logical consequence and validity:

$$\boxed{F \models G \quad \text{iff} \quad \models F \Rightarrow G}$$

What is nonmonotonic logic?

In nonmonotonic logics, conclusions don't behave monotonically – if information is added to the knowledge base, it might happen that previous conclusions are given up, like in the famous Tweety example:

Tweety the penguin

Birds fly, penguins are birds, but penguins don't fly

$$bird \mathrel{|\!\sim} fly, penguin \wedge bird \mathrel{|\!\sim} \neg fly$$

Why nonmonotonic logic?

Nonmonotonic reasoning is indispensable for applications dealing with uncertain, incomplete information and should better be termed rational commonsense reasoning:

Nonmonotonic inference . . .

. . . "*is not to add certain knowledge where there is none, but rather to guide the selection of tentatively held beliefs in the hope that fruitful investigations and good guesses will result.*"

D. McDermott & J. Doyle, *Nonmonotonic logic*, 1980

The relevance of uncertain reasoning

Many applications today use classical logic or even weaker logics[1], but ...

Certainty is a treacherous illusion!

- Crucial and popular strategies of classical logics do not hold for uncertain reasoning: Modus ponens, contraposition, transitivity/syllogism, monotony, ...
- Inconsistencies and contradictions can not be resolved.

Costly or even disastrous consequences may result from ignoring uncertainty.

[1] E.g., for business rules often production rule engines are used.

A word on Tweety and penguins

The famous Tweety example deals with the important subclass-superclass-problem, like in this (less funny) example:

Example – Cancer

Cancer patients are usually adults.
Neuroblastoma is a form of cancer.
Lena is suffering from neuroblastoma.

Lena is 1 year old.[a]

[a] Neuroblastoma occurs (basically) only in children and is here the most frequent cancer disease with solid tumors.

Tweety and penguins – intuitive example that allows immediate approvement or rejection of conclusions by active reasoners (without making them feel unhappy).

Logical consequence vs. (uncertain) human inference

Logical consequence $\models$ $\quad \mathcal{F} \models \mathcal{G}$ iff $Mod(\mathcal{F}) \subseteq Mod(\mathcal{G})$

Consequence operator

$$\begin{aligned} Cn &: 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}} \\ Cn(\mathcal{F}) &= \{G \in \mathcal{L} \mid \mathcal{F} \models G\} \end{aligned}$$

$$\boxed{\mathcal{F} \models \mathcal{G} \text{ gdw. } \mathcal{G} \subseteq Cn(\mathcal{F})}$$

Commonsense reasoning $\mid\!\sim$?????

Inference operator

$$\begin{aligned} C &: 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}} \\ C(\mathcal{F}) &= \{G \in \mathcal{L} \mid \mathcal{F} \mid\!\sim G\} \end{aligned}$$

$$\boxed{\mathcal{F} \mid\!\sim \mathcal{G} \text{ gdw. } \mathcal{G} \subseteq C(\mathcal{F})}$$

Monotony

$$\mathcal{F} \subseteq \mathcal{H} \text{ implies } Cn(\mathcal{F}) \subseteq Cn(\mathcal{H})$$

- Cn considers all models.
- Also Transitivity and Contraposition are based (in principle) on Monotony.
- Monotony does not allow to revise inferences.

Expect a defeasible inference operation C to be nonmonotonic!

Talk Overview

Section 3

Formal models of commonsense reasoning

Basic strategies of (nonmonotonic) commonsense reasoning

Like in classical logic, and although Modus Ponens is invalid in general,

rules

are the main carriers of nonmonotonic inference. However, syntax and/or semantics of rules are different from implications in classical logic.

Basically, two types of rules are used:

- Rules with default assumptions: Reiter's default logic, answer set programming, weak completion semantics
- Defeasible rules: Conditional reasoning, Poole's default logic

Reiter's default rules

Let φ, $\psi_1, \ldots, \psi_n$ and χ be (classical) formulas.

Reiter default rule

$$\delta = \frac{\varphi \;:\; \psi_1, \ldots, \psi_n}{\chi}$$

with the reading

If φ is known, and $\psi_1, \ldots, \psi_n$ can be consistently assumed (i.e., none of $\neg\psi_i$ is known), then conclude χ.

$\varphi = pre(\delta)$	Precondition
$\chi = cons(\delta)$	Consequence
$\{\psi_1, \ldots, \psi_n\} = just(\delta)$	Justifications

Reiter default theory (W, Δ): W classical formulas, Δ defaults.
Semantics is given by extensions which are minimal sets of classical formulas closed under deduction and default application.

Example: Suppression task [Byrne 1989]

In an empirical study [Byrne 1989]

(α) *If she has an essay to write, then she will study late in the library* and — $(e \rightarrow l)$

(γ) *If the library stays open, she will study late in the library* and — $(o \rightarrow l)$

(δ) *She has an essay to write.* — (e)

only 38 % of the participants make a modus ponens inference and conclude that: *She will study late in the library*.
62% concluded that: *She may or may not study late in the library.*

Can this be modelled via Reiter?

Reiter and Suppression task [Byrne 1989; Ragni, Eichhorn, Kern-Isberner 2016]

A suitable Reiter default theory for modelling this problem would be

$T_{supp} = (W_{supp}, \Delta_{supp})$ with $W_{supp} = \{e\}$ and

$$\Delta_{supp} = \left\{ \delta_1 = \frac{e : \overline{\mathsf{ab}_1}}{l};\ \delta_2 = \frac{l : o}{o};\ \delta_3 = \frac{\overline{o} : \mathsf{ab}_1}{\mathsf{ab}_1} \right\}$$

where ab_1 is an *abnormality predicate* which expresses that nothing abnormal is known.

Extensions via process trees – definitions [Antoniou 1997]

- δ is applicable to a deductively closed set $\mathcal{F}$ iff $pre(\delta) \in \mathcal{F}$ and $\neg B \notin \mathcal{F}$ for every $B \in just(\delta)$.
- (default) process $\Pi = (\delta_1, ..., \delta_m)$: finite sequence of defaults $\delta_i \in \Delta$ with the two sets

$$\begin{aligned} In(\Pi) &= Cn(W \cup \{cons(\delta) | \delta \in \Pi\}) \\ Out(\Pi) &= \{\neg A | A \in just(\delta), \delta \in \Pi\} \end{aligned}$$

  such that each δ is applicable to the In-set of the foregoing defaults.
- Π is
  - successful iff $In(\Pi) \cap Out(\Pi) = \varnothing$,
  - closed iff every $\delta \in \Delta$ that is applicable to $In(\Pi)$ is an element of Π.
- $\mathcal{E}$ is an *extension* of (W, D) iff $\mathcal{E} = In(\Pi)$ for a closed and successful process Π.

Process tree: Suppression task [Ragni, Eichhorn, Kern-Isberner 2016]

$$\delta_1 = \frac{e : \overline{\mathsf{ab}_1}}{l}$$

$$\delta_2 = \frac{l : o}{o}$$

$$\begin{array}{rcl} Cn(\{e\}) & \bullet & \varnothing \\ & \downarrow \delta_1 & \\ Cn(\{e, l\}) & \bullet & \{\mathsf{ab}_1\} \\ & \downarrow \delta_2 & \\ Cn(\{e, l, o\}) & \bullet & \{\mathsf{ab}_1, \neg o\} \end{array}$$

$Cn(\{e, l, o\})$ is the only extension of this theory, i.e.,
Modus ponens cannot be suppressed.

Inference relation for default logics

Let (W, Δ) be a default theory.
A classical formula ϕ follows nonmonotonically from W by exploiting Δ,

$$W \quad \mid\!\sim_{\Delta}^{Reiter} \quad \phi$$

if ϕ is contained in all extensions of (W, Δ).

$$C_{\Delta}^{Reiter}(W) = \{\phi \mid W \mid\!\sim_{\Delta}^{Reiter} \phi\}$$

is the corresponding inference operator.

In the suppression task example, we have

$$e \mid\!\sim_{\Delta}^{Reiter} l,$$

so no suppression effect occurs.

Logic programming

In logic programming, the (commonsense) implication “if she has an essay to finish, she will study late in the library” should be encoded by the clause

$$l \leftarrow e \wedge \overline{\mathsf{ab}_1}$$

The so-called weak-completion semantics [Hölldobler and Kencana Ramli, 2009] works as follows:

1. Replace all clauses with the same head by a disjunction of the body elements, i.e., $A \leftarrow B_1, \ldots, A \leftarrow B_n$ by $A \leftarrow B_1 \vee \ldots \vee B_n$.
2. Replace all occurrences of $\leftarrow$ by $\leftrightarrow$.

Logic Programming: Suppression task [Dietz et al., 2012]

Program	$l \leftarrow e \wedge \overline{\mathsf{ab}}_1$
	$l \leftarrow o \wedge \overline{\mathsf{ab}}_3$
	$\mathsf{ab}_1 \leftarrow \overline{o}$
	$\mathsf{ab}_3 \leftarrow \overline{e}$
	$e \leftarrow \top$
wcs	$l \leftrightarrow (e \wedge \overline{\mathsf{ab}}_1) \vee (o \wedge \overline{\mathsf{ab}}_3)$
	$\mathsf{ab}_1 \leftrightarrow \overline{o}$
	$\mathsf{ab}_3 \leftrightarrow \overline{e}$
	$e \leftrightarrow \top$
Least Model	$(\{e\}, \{\mathsf{ab}_3\})$

Weak completion semantics can model the suppression effect.

Defeasible rules and conditionals

Defeasible rules establish an uncertain, defeasible connection between antecedent A and consequent B of a rule and can be (logically) implemented by conditionals

$(B|A)$ – "If A then (usually, probably, plausibly ...) B"

- Conditionals encode semantical relationships (plausible inferences) between the antecedent A and the consequent B.
- Conditionals implement nonmonotonic inferences via "$(B|A)$ is accepted iff $A \mid\!\sim B$ holds".
- Conditionals occur in different shapes in many approaches (e.g., as conditional probabilities in Bayesian approaches),
- Conditionals seem to be similar to classical (material) implications "If A then (definitely) B", but are substantially different!

Indeed, many fallacies observed when applying classical logic to uncertain domains are caused by mixing up implications and conditionals!

Conditionals and implications – example

Christmas on the northern hemisphere

- If Christmas were in summer, there would be no snow at Christmas.
  plausible, approved
- If Christmas were in summer, there would be no Christmas gifts.
  strange, why?
- If Christmas were in summer, there would be no gravitation.
  downright nonsense!

All these statements are logically true, when understood as (material) implications (because Christmas is in winter on the northern hemisphere, hence the antecedent is false!).

However, understood as conditionals, crucial differences appear!

What makes conditionals so special?

A conditional $(B|A)$ focusses on cases where the premise A is fulfilled but does not say anything about cases when A does not hold – conditionals go beyond classical logic, as they are three-valued entities.

A conditional leaves more semantical room for modelling acceptance in case its confirmation $A \wedge B$ is more plausible than its refutation $A \wedge \neg B$.

Conditional acceptance and preferential entailment $\mid\!\sim_{\prec}$ [Makinson 89]

Let $\prec$ be a (well-behaved) relation on models (expressing , e.g., plausibility).

$$(B|A) \text{ is accepted} \quad \text{iff} \quad A \mid\!\sim_{\prec} B$$

iff in the most plausible models of A (wrt $\prec$), B holds also.

$\mid\!\sim_{\prec}$ is a semantic-based nonmonotonic inference relation that is encoded by conditionals on the syntax level.

Ranking functions and conditionals

Ordinal conditional functions (OCF, ranking functions[2]) [Spohn 1988]

$\kappa : \Omega \to \mathbb{N}(\cup\{\infty\})$ (Ω set of possible worlds, $\kappa^{-1}(0) \neq \varnothing$)

$\kappa(\omega_1) < \kappa(\omega_2)$	ω_1 is more plausible than ω_2
$\kappa(\omega) = 0$	ω is maximally plausible
$\kappa(A)$	$:= \min\{\kappa(\omega) \mid \omega \models A\}$
Bel (κ)	$:= \{A \mid \kappa(\neg A) > 0\}$

Validating conditionals

$$\kappa \models (B|A) \text{ iff } \kappa(AB) < \kappa(A\overline{B})$$

κ accepts a conditional $(B|A)$ iff its verification AB is more plausible than its falsification $A\overline{B}$.

[2] Rankings can be understood as qualitative abstractions of probabilities

Ranking functions – example

Example (ranked flyers)

$\kappa(\omega) = 4$	$p\bar{b}\,f$
$\kappa(\omega) = 2$	$pbf \quad p\bar{b}\,\overline{f}$
$\kappa(\omega) = 1$	$pb\overline{f} \quad \overline{p}\,b\overline{f}$
$\kappa(\omega) = 0$	$\overline{p}\,bf \quad \overline{p}\,\overline{b}\,f \quad \overline{p}\,\overline{b}\,\overline{f}$

$$Bel\,(\kappa) = Cn(\overline{p}\,(f \vee \overline{b}\,\overline{f}\,)$$

$$\kappa(bf) = 0 < 1 = \kappa(b\overline{f}\,) \Longrightarrow \kappa \models (f|b),$$

but $\kappa(p\overline{f} \; = 1 < 2 = \kappa(pf) \Longrightarrow \kappa \models (\overline{f}\,|p)$

(also $\kappa \models (b|p)$)

Verification und falsification of conditionals

$\omega \in \Omega$ a possible world, $(B|A)$ a conditional

- ω verifies $(B|A)$iff $\omega \models AB$;
- ω falsifies $(B|A)$ iff $\omega \models A\overline{B}$;
- ω satisfies $(B|A)$ iff $\omega \models A \Rightarrow B$ (classical counterpart to $(B|A)$).

Verification implies satisfaction.

Conditionals – example [Goldszmidt & Pearl 1996]

Conditional knowledge base Δ:

$r_1 : (f\|b)$	*birds fly*
$r_2 : (b\|p)$	*penguins are birds*
$r_3 : (\overline{f}\|p)$	*penguins don't fly*
$r_4 : (w\|b)$	*birds have wings*
$r_5 : (a\|f)$	*animals that fly are airborne*

$\omega = pb\overline{f}w\overline{a}$

- ω verifies r_2, r_3, r_4,
- ω falsifies r_1,
- ω satisfies r_2, r_3, r_4, r_5.

Conditionals and tolerance

A conditional knowledge base Δ is consistent iff there is κ such that $\kappa \models \Delta$.

Goal: Finding a simple criterion to decide whether Δ is consistent or not.

$(B|A)$ is tolerated by $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, if there is $\omega \in \Omega$ such that

$$\omega \models AB \wedge \bigwedge_{i=1}^{n} (A_i \Rightarrow B_i),$$

i.e., if there is $\omega \in \Omega$ that verifies $(B|A)$ and satisfies all conditionals in Δ.

Tolerance – example

$\Delta = \{r_1 : (f|b),\ r_2 : (b|p),\ r_3 : (\overline{f}|p),\ r_4 : (w|b),\ r_5 : (a|f)\}$

r_1 is tolerated by Δ:

E.g.,

$$\omega = \overline{p}bfwa \models bf \wedge (p \Rightarrow b) \wedge (p \Rightarrow \overline{f}) \wedge (b \Rightarrow w) \wedge (f \Rightarrow a)$$

Likewise, r_4 and r_5 are tolerated by Δ.

However, r_2 is not tolerated by Δ because

$$pb \wedge (b \Rightarrow f) \wedge (p \Rightarrow \overline{f}) \wedge (b \Rightarrow w) \wedge (f \Rightarrow a) \equiv \bot$$

Likewise, r_3 is not tolerated by Δ.

Consistent conditional knowledge bases

Theorem (Adams 1975)

A conditional knowledge Δ is consistent iff each (non-empty) subset $\Delta' \subseteq \Delta$ contains a conditional that is tolerated by Δ'.

This implies

Theorem

Δ is consistent iff there is a partition $\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_k)$ such that each conditional in Δ_i is tolerated by $\cup_{j=i}^{k} \Delta_j$.

Consistency Test Algorithm [Pearl 1990]

Input : Conditional knowledge base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$;
Output : Partition $\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_k)$ (as described above) iff Δ is consistent.

1. Set $i := 0$.
2. **While** $\Delta \neq \varnothing$
   1. Find the subset Δ_i consisting of all conditionals in Δ that are tolerated by Δ;
   2. if there is no such conditional then **Halt**: Δ is inconsistent;
   3. Else set $\Delta := \Delta - \Delta_i$, $i := i + 1$;
3. **Return** $\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_k)$

Complexity: $O(n^2)$ SAT-tests in 2.1.

Consistency test and partitioning – example

$\Delta = \{r_1 : (f|b), r_2 : (b|p), r_3 : (\overline{f}|p), r_4 : (w|b), r_5 : (a|f)\}$

$\Delta_0 = \{r_1, r_4, r_5\}$ because r_1, r_4, r_5 are tolerated by all conditionals in Δ, but r_2 and r_3 are not.

r_2 and r_3 tolerate each other, hence $\Delta_1 = \{r_2, r_3\}$.
Therefore, we obtain

$$\Delta = (\Delta_0, \Delta_1)$$

System Z [Pearl 1990]

We use the partitioning of $\Delta = \{r_i : (B_i|A_i)\}_{1 \leqslant i \leqslant n}$ to define a "best" κ that is a model of Δ:

$$\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_k),$$

induces a ranking Z of the conditionals $r_i = (B_i|A_i) \in \Delta$:

$$Z(r_i) = j \quad \text{iff } r_i \in \Delta_j;$$

this gives us the ranking function κ^z defined by

$$\kappa^z(\omega) = \begin{cases} 0, & \text{if } \omega \text{ does not falsify any conditional in } \Delta \text{ ,} \\ \max_{1 \leqslant i \leqslant n}\{Z(r_i) \mid \omega \models A_i\overline{B_i}\} + 1, & \text{otherwise} \end{cases}$$

κ^z imposes penalty points on the worlds for falsifying conditionals.

System Z (cont'd)

Theorem (System Z)

κ^z is a model of Δ, i.e., $\kappa^z \models \Delta$, and is minimal among all models of Δ, i.e., for all other κ such that $\kappa \models \Delta$, there is at least one $\omega \in \Omega$ with $\kappa(\omega) > \kappa^z(\omega)$.

κ^z implements maximal plausibility among all models of Δ. .

Z-inference wrt Δ, $\mid\!\sim^z_\Delta$, is defined as follows:

$$A \mid\!\sim^z_\Delta B \quad \text{iff} \quad \kappa^z(AB) < \kappa^z(A\overline{B})$$

$\mid\!\sim^z_\Delta$ is one of the best existing nonmonotonic inference systems.

System Z – example 1

$\Delta = \{(f|b),(b|p),(\overline{f}|p)\}$ with the following partitioning:

$$\Delta_0 = \{(f|b)\}, \quad \Delta_1 = \{(b|p),(\overline{f}|p)\},$$

hence $Z(f|b) = 0$ and $Z(b|p) = Z(\overline{f}|p) = 1$.

ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$
pbf	2	$\overline{p}bf$	0
$pb\overline{f}$	1	$\overline{p}b\overline{f}$	1
$p\overline{b}f$	2	$\overline{p}\overline{b}f$	0
$p\overline{b}\,\overline{f}$	2	$\overline{p}\overline{b}\,\overline{f}$	0

$$pb \mid\!\sim^z_\Delta \overline{f}$$

System Z – example 2

$\Delta = \{r_1 : (f|b), r_2 : (b|p), r_3 : (\overline{f}|p), r_4 : (w|b), r_5 : (a|f)\}$

Partitioning:

$$\begin{aligned} \Delta_0 &= \{r_1, r_4, r_5\} \\ \Delta_1 &= \{r_2, r_3\}, \end{aligned}$$

therefore

$$\begin{aligned} Z(r_1) &= Z(r_4) = Z(r_5) = 0, \\ Z(r_2) &= Z(r_3) = 1. \end{aligned}$$

System Z – example 2 (cont'd)

ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$
$pbfwa$	2	$pbfw\overline{a}$	2	$pbf\overline{w}a$	2	$pbf\overline{w}\,\overline{a}$	2
$pb\overline{f}wa$	1	$pb\overline{f}w\overline{a}$	1	$pb\overline{f}\overline{w}a$	1	$pb\overline{f}\overline{w}\,\overline{a}$	1
$p\overline{b}fwa$	2	$p\overline{b}fw\overline{a}$	2	$p\overline{b}f\overline{w}a$	2	$p\overline{b}f\overline{w}\,\overline{a}$	2
$p\overline{b}\,\overline{f}wa$	2	$p\overline{b}\,\overline{f}w\overline{a}$	2	$p\overline{b}\,\overline{f}\overline{w}a$	2	$p\overline{b}\,\overline{f}\overline{w}\,\overline{a}$	2
$\overline{p}bfwa$	0	$\overline{p}bfw\overline{a}$	1	$\overline{p}bf\overline{w}a$	1	$\overline{p}bf\overline{w}\,\overline{a}$	1
$\overline{p}b\overline{f}wa$	1	$\overline{p}b\overline{f}w\overline{a}$	1	$\overline{p}b\overline{f}\overline{w}a$	1	$\overline{p}b\overline{f}\overline{w}\,\overline{a}$	1
$\overline{p}\overline{b}fwa$	0	$\overline{p}\overline{b}fw\overline{a}$	1	$\overline{p}\overline{b}f\overline{w}a$	0	$\overline{p}\overline{b}f\overline{w}\,\overline{a}$	1
$\overline{p}\overline{b}\,\overline{f}wa$	0	$\overline{p}\overline{b}\,\overline{f}w\overline{a}$	0	$\overline{p}\overline{b}\,\overline{f}\overline{w}a$	0	$\overline{p}\overline{b}\,\overline{f}\overline{w}\,\overline{a}$	0

$b \mid\!\sim^{z}_{\Delta} a$ because $\kappa^z(ba) = 0 < 1 = \kappa^z(b\overline{a})$.

Drowning problem for system Z

Δ: $r_1 : (f|b)$ *Birds fly*
$r_2 : (b|p)$ *Penguins are birds*
$r_3 : (\overline{f}|p)$ *Penguins don't fly*
$r_4 : (w|b)$ *Birds have wings*

Do penguins (as non-typical birds) wings?

Z-partitioning:

$$\Delta_0 = \{r_1, r_4\},\ \Delta_1 = \{r_2, r_3\};$$

yielding the system-Z representation κ_z:

Drowning problem for system Z (cont'd)

ω	r_i fals.	$\kappa_z(\omega)$	ω	r_i fals.	$\kappa_z(\omega)$
$pbfw$	r_3	2	$\overline{p}bfw$	–	0
$pbf\overline{w}$	r_3, r_4	2	$\overline{p}bf\overline{w}$	r_4	1
$pb\overline{f}w$	r_1	1	$\overline{p}b\overline{f}w$	r_1	1
$pb\overline{f}\overline{w}$	r_1, r_4	1	$\overline{p}b\overline{f}\overline{w}$	r_1, r_4	1
$p\overline{b}fw$	r_2, r_3	2	$\overline{p}\overline{b}fw$	–	0
$p\overline{b}f\overline{w}$	r_2, r_3	2	$\overline{p}\overline{b}f\overline{w}$	–	0
$p\overline{b}\,\overline{f}w$	r_2	2	$\overline{p}\overline{b}\,\overline{f}w$	–	0
$p\overline{b}\,\overline{f}\overline{w}$	r_2	2	$\overline{p}\overline{b}\,\overline{f}\overline{w}$	–	0

$\kappa_z(pw) = 1 = \kappa_z(p\overline{w})$ – we cannot decide if penguins have wings or not

$\rightarrow$ Drowning problem: In $pb\overline{f}\overline{w}$, two conditionals with the same Z-rank are falsified, hence one of them "drowns".

C-representations [Kern-Isberner 2001]

An alternative to system Z: $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$

c-representation of Δ is defined by

$$\kappa_\Delta(\omega) = \sum_{\omega \models A_i \overline{B_i}} \kappa_i^-$$

with parameters $\kappa_1^-, \ldots, \kappa_n^- \in \mathbb{N}_0$ chosen such that

$$\kappa_\Delta \models (B_j|A_j), 1 \leqslant j \leqslant n,$$

holds, i.e.,

$$\kappa_j^- > \min_{\omega \models A_j B_j} \sum_{\substack{i \neq j \\ \omega \models A_i \overline{B_i}}} \kappa_i^- - \min_{\omega \models A_j \overline{B_j}} \sum_{\substack{i \neq j \\ \omega \models A_i \overline{B_i}}} \kappa_i^-$$

C-representations and the Drowning problem

Considering again:

Δ: $r_1 : (f|b)$ *Birds fly*
$r_2 : (b|p)$ *Penguins are birds*
$r_3 : (\overline{f}|p)$ *Penguins don't fly*
$r_4 : (w|b)$ *Birds have wings*

Compute (pareto-)minimal parameters κ_i^- for each conditional:

$$\kappa_1^- = \kappa_4^- = 1,\ \kappa_2^- = \kappa_3^- = 2;$$

minimal c-representation κ_Δ:

C-representations and the Drowning problem (cont'd)

ω	r_i fals.	$\kappa_\Delta(\omega)$	ω	r_i fals.	$\kappa_\Delta(\omega)$
$pbfw$	r_3	2	$\overline{p}bfw$	–	0
$pbf\overline{w}$	r_3, r_4	3	$\overline{p}bf\overline{w}$	r_4	1
$pb\overline{f}w$	r_1	1	$\overline{p}b\overline{f}w$	r_1	1
$pb\overline{f}\overline{w}$	r_1, r_4	2	$\overline{p}b\overline{f}\overline{w}$	r_1, r_4	2
$p\overline{b}fw$	r_2, r_3	4	$\overline{p}\overline{b}fw$	–	0
$p\overline{b}f\overline{w}$	r_2, r_3	4	$\overline{p}\overline{b}f\overline{w}$	–	0
$p\overline{b}\,\overline{f}w$	r_2	2	$\overline{p}\overline{b}\overline{f}w$	–	0
$p\overline{b}\,\overline{f}\overline{w}$	r_2	2	$\overline{p}\overline{b}\,\overline{f}\overline{w}$	–	0

Here we have $\kappa_\Delta(pw) = 1 < 2 = \kappa_\Delta(p\overline{w})$ – hence we can infer that penguins have wings.

OCF inference and Suppression effect

Coming back to the Suppression effect:

$$KB_1 = \{\delta_1 : (l|e), \delta_2 : (l|o), \delta_3 : (e|\top)\}$$
$$KB_2 = \{\delta_1 : (l|e), \delta_4 : (o|l), \delta_3 : (e|\top)\}$$

$\Delta_0 = KB_i$ in each case, so $Bel(\kappa_1^z) = Cn(el)$ and $Bel(\kappa_2^z) = Cn(elo)$, hence l is believed in both cases and therefore, no suppression effect occurs with system Z.

C-representations behave very similar as system Z in this example and don't show the suppression effect either.

System Z vs. c-representations

ω	$e\,l\,o$	$e\,l\,\overline{o}$	$e\,\overline{l}\,o$	$e\,\overline{l}\,\overline{o}$	$\overline{e}\,l\,o$	$\overline{e}\,l\,\overline{o}$	$\overline{e}\,\overline{l}\,o$	$\overline{e}\,\overline{l}\,\overline{o}$
$\kappa_1^Z(\omega)$	0	0	1	1	1	1	1	1
$\kappa_1^c(\omega)$	0	1	1	1	1	1	2	1
$\kappa_2^Z(\omega)$	0	1	1	1	1	1	1	1
$\kappa_2^c(\omega)$	0	1	2	1	1	2	1	1

Mimicking weak completion semantics

In the modelling by logic programs, a strong connection is established between l and both o and e.

So, if we encode this in the knowledge base

$$\Delta_3 = \{(l|eo), (e|\top)\},$$

then we find both for system Z and c-representations that

$$Bel(\kappa_3^z) = Bel(\kappa_3^c) = Cn(elo \vee e\bar{l}\bar{o}),$$

which means that l is no longer believed!

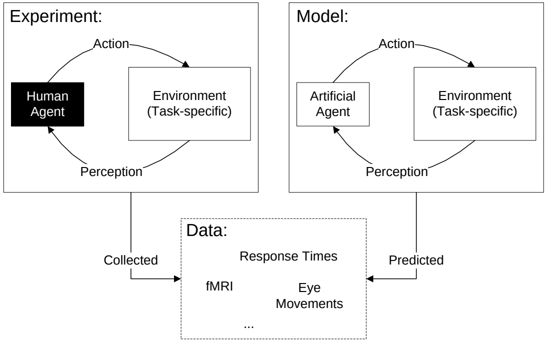
Occurrence of the suppression effect depends more on the modelling than on the chosen method!

Talk Overview

Section 4

Cognitive aspects of Cognitive Logics

What does a cognitive model?



- Reconstructive and generative models (Lüer & Spada, 1990):
  - Reconstructive: Conceptualising structures and processes that underly mental activity
  - Generative: The execution of a model not only describes psychological phenomena but also generates them
    $\Rightarrow$ Compare model predictions with empirical data

How can we evaluate cognitive theories?

Simon and Wallach (1999) require a generative theories to have:

- Product correspondence: this requires that the cognitive model shows a similar overall performance as human data
- Correspondence of intermediate steps: this requires that assumed processes and steps in the model parallels separable stages in human processing
- Error correspondence: this requires that the same error patterns in the model emerge than in experimental data
- Correspondence of context dependency: this is a comparable sensitivity to known external influences

Requirements of cognitively-adequate logics

- Explainability
- Prediction of intermediate steps
- Reverse engineering
- Implementability
- Product correspondence, i.e., same inferences
- Embedding "fallacies" in a logical context
- Alignment of cognitive and logical theories

What phases of cognitive modeling exist?

Four phases can be considered (e.g., Lewandowski & Farrell, 2011):

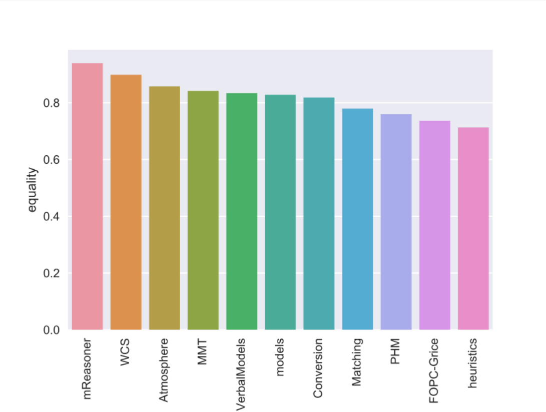
1. Task analysis:
    - What knowledge is needed to solve a task?
    - What are processes involved in generating the knowledge to solve a task
    - What are relevant structures an architecture used to specify a model?
2. Empirical data
    - Reconstruction of trace/statistical measure for one participant
    - Reconstruction of some statistical measure which considers all participants

What phases of cognitive modeling exist?

3. Model implementation
   - Architecture selection (e.g. Neural Network, MPT, Logic)
   - Process specification
   - Parameter estimation (e.g. simulated annealing, maximum likelihood estimation)
4. Model validation
   - Parameter uncertainty
   - Model comparison
   - Model interpretation

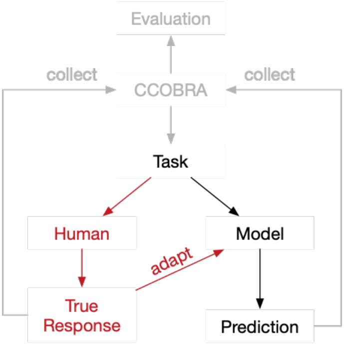
$\Rightarrow$ Mental representation ($\rightarrow$ conditionals) and the inference mechanism are core issues

Syllogistic Reasoning: Aggegate Data



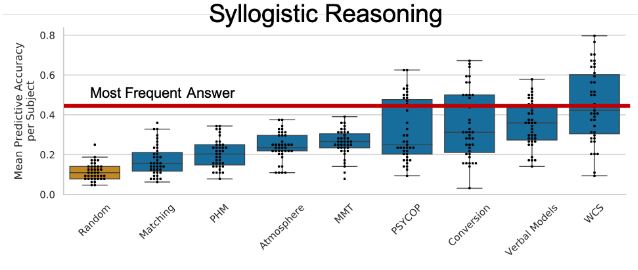
- Existing cognitive theories do reach a high predictive power for aggregate data, i.e., predicting distribution of answers
- But, if we want to have an AI assistant that can adapt to our reasoning capabilities, does modeling 'group answers" really helps us?

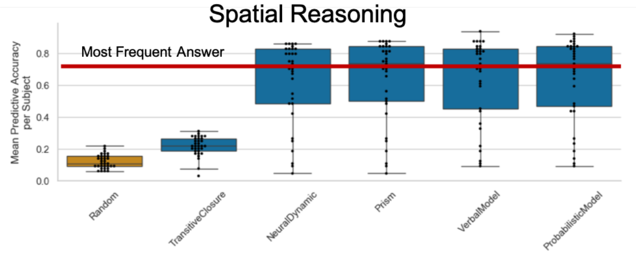
Predict the Individual Reasoner



- Model receives general training problems
- Framework presents task
- Model generates predictions
- Prediction compared with true response
- Model adapts to the human response
- Framework presents next task

Predict the Individual Reasoner

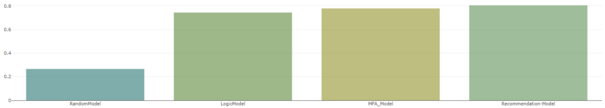




Predict the Individual Reasoner: Propositional

Accuracy

Computes the accuracy of the models, i.e., the percentage of correct predictions.



SubjectBoxes

The following plot depicts boxplots for the models indicating individual subject performance. The data used for the plot are accuracies for individuals. Consequently, min and max refer to the accuracy of the worst and best matching subjects. The dots refer to the mean accuracies of individual participants.

Commonsense inference rules

From a conditional statement "If A then B",
Modus ponens and Modus tollens are logically valid inference rules:

(MP) From A, infer B

(MT) From $\neg B$, infer $\neg A$

However, people also use other inference rules in commonsense reasoning:

(AC) Affirmation of the Consequent: From B, infer A

(DA) Denial of the Antecedent: From $\neg A$, infer $\neg B$

Logical invalidity in the Suppression Task

In the Suppression Task [Byrne 1989], participants had to draw inferences with respect to the arguments

Suppression Task

"If Lisa has an essay to write, she will study late in the library."
"If the library stays open, she will study late in the library."
"Lisa has an essay to write."

Here, the majority of the participants (students without tuition in logic)

- did not apply MP (38%) nor MT (33%),
- but did apply AC (63%) and DA (54%).

Applying AC and DA is usually deemed to be irrational, i.e., rationality is usually assessed according to classical logic.

Inference patterns

However, people deviate so systematically from (MP) and (MT) and apply so frequently (AC) and (DA) that commonsense logics have to find a model for this.

[Eichhorn, Kern-Isberner & Ragni AAAI-2018]

Using a (nonmonotonic) conditional logic as normative theory to evaluate human inferences eliminates (basically) all irrationality!

Basic idea: Consider all four inference rules (MP, MT, AC, DA) together in a 4-tuple to model generic inference behaviour:

Definition

An inference pattern ϱ is a 4-tuple that for each inference rule MP, MT, AC, and DA indicates whether the rule is used (positive rule, e.g., MP) or not used (negated rule, e.g., $\neg\mathrm{MP}$) in an inference scenario.

Inference patterns – examples

- Suppression Task: (MP (38%), MT (33%), AC (63%), DA (54%)) yields the inference pattern $\varrho_{B89} = (\neg\mathrm{MP}, \neg\mathrm{MT}, \mathrm{AC}, \mathrm{DA})$.
- Counterfactuals [Thompson &Byrne 2002]: "If the car had been out of $\underline{g}$as, then it would have $\underline{s}$talled."
  Overall inferences: (MP (78%), MT (85%), AC (41%), DA (50%)), yielding the inference pattern $\varrho_{TB02} = (\mathrm{MP}, \mathrm{MT}, \neg\mathrm{AC}, \mathrm{DA})$.

Sensitivity of inference behavior

Different wordings and slightly different information can change human inferences drastically –

- What do people understand from the reasoning task?
  $\rightarrow$ implicit assumptions, background knowledge
- Additional information may suggest implicitly exceptions, alternatives, strengthening etc
  $\rightarrow$ nonmonotonic reasoning
- "If . . . then"-statements often are not strict
  $\rightarrow$ conditionals

$\rightarrow$ Basics of nonmonotonic logics

Remember the basics of nonomotonic logics and plausibility:

Total preorders $\preccurlyeq$ on possible worlds expressing plausibility are of crucial importance both for nonmonotonic reasoning and conditionals:

$\omega_1 \preccurlyeq \omega_2$	ω_1 is deemed at least as plausible as ω_2
$A \preccurlyeq B$	iff minimal models of A are at least as plausible as all models of B
$A \mid\!\sim_{\preccurlyeq} B$	iff $AB \prec A\overline{B}$ – in the context of A, B is more plausible than $\overline{B}$
Ψ	epistemic state equipped with a total preorder $\preccurlyeq_\Psi$
$Bel(\Psi)$	$= Th(\min(\preccurlyeq_\Psi))$ most plausible beliefs in Ψ

Inference patterns → plausibility constraints

Rule	Inference	Plaus. constraint
MP	$A \mid\!\sim B$	$A\,B \prec A\,\overline{B}$
MT	$\overline{B} \mid\!\sim \overline{A}$	$\overline{A}\,\overline{B} \prec A\,\overline{B}$
AC	$B \mid\!\sim A$	$AB \prec \overline{A}\,B$
DA	$\overline{A} \mid\!\sim \overline{B}$	$\overline{A}\,\overline{B} \prec \overline{A}\,B$

Negated inference rules (e.g., ¬MP) are implemented simply by negating the constraint (e.g., $A\,\overline{B} \preccurlyeq A\,B$).

Rationality in terms of nonmonotonic logics

reasoning pattern $\varrho \longrightarrow$ set of plausibility constraints $\mathcal{C}(\varrho)$

$\mathcal{C}(\varrho)$ is satisfiable iff there is a plausibility relation $\preceq$ on possible worlds that satisfies all constraints in $\mathcal{C}(\varrho)$

- An inference pattern $\varrho \in \mathcal{R}$ is called rational iff there is a plausibility relation $\prec$ that satisfies $\mathcal{C}(\varrho)$.
- Otherwise, the inference pattern is irrational.

In over 60 empirical studies investigated so far, hardly any irrational patterns could be found (less than 2%).

Only 2 out of 16 patterns are irrational.

Empirical study – overview

22 studies with 35 experiments [Spiegel TU Dortmund 2018] –

Only six inference patterns were ever drawn at a frequency of more than 5%. The proportion of irrational patterns is only 1.1%.

Most frequent inference patterns:

(MP, MT, AC, DA)	perc.	meaning
TTTT	33.9	"credulous reasoner"
TTFF	23.6	"the logical reasoner"
TTTF	12.1	"partly logical reasoner"
TFTF	9.2	"reasoner rejecting negations"
TFTT	5.7	"bold reasoner" (all but MT)
TFFF	5.7	"basic reasoner (only MP)

(For more on this: see our talk on Thursday morning)

Example counterfactuals (cont'd)

Constraints for the inference pattern $\varrho_{TB02} = (\mathrm{MP}, \mathrm{MT}, \neg\mathrm{AC}, \mathrm{DA})$:

$$\frac{\{AB \prec A\overline{B}, \overline{A}\,\overline{B} \prec A\overline{B}, \overline{A}B \preccurlyeq AB, \overline{A}\,\overline{B} \prec \overline{A}B\}}{\equiv \quad \overline{A}\,\overline{B} \prec \overline{A}B \preccurlyeq AB \prec A\overline{B}}$$

In this example, $\mathit{Bel}(\varrho_{TB02}) = \mathit{Cn}(\overline{A}\,\overline{B})$.

$\rightarrow$ Finding: In the counterfactual case, people believe not only that the antecedent is false[3], but also that the consequent is false!

[3] This is usually assumed in the counterfactual case

Background knowledge

Using c-representations and their parameters κ_i^-, we can further elaborate on the background knowledge that people (may) have used for reasoning:

With the algorithm Explanation generator [Eichhorn, Kern-Isberner, Ragni 2018] we're able to extract basic conditionals from inference patterns according to the following schema:

Rule	Conditional	Rule	Conditional
MP	$(B\|A)$	¬MP	$(\!(\overline{B}\|A)\!)$
MT	$(\overline{A}\|\overline{B})$	¬MT	$(\!(A\|\overline{B})\!)$
AC	$(A\|B)$	¬AC	$(\!(\overline{A}\|B)\!)$
DA	$(\overline{B}\|\overline{A})$	¬DA	$(\!(B\|\overline{A})\!)$

Strong and weak conditionals for the inference rules

Background knowledge in the Suppression task

Here we have the inference pattern

$$\varrho_{B89} = (\neg \mathrm{MP}, \neg \mathrm{MT}, \mathrm{AC}, \mathrm{DA}).$$

Explanation generator $\rightarrow$ two KBs can explain the inference pattern:

- $KB_{B89} = \{(e|l)\}$
  "If Lisa is in the library, then she (usually) has an essay to write"
- $KB'_{B89} = \{(\bar{l}|\bar{e})\}$
  "If Lisa does not have an essay to write, then she (usually) is not in the library"

This also explains the rationality of the inference pattern:

Participants might have understood the given conditional information in its inverse form, and hence applied AC and DA which, in fact, amount to MP and MT for the inverse conditional.

Preliminary summary

Logics based on conditionals and rankings/total preorders can provide/ensure

- Explainability ✓
- Reverse engineering of human inferences ✓
- Implementability ✓
- Product correspondence, i.e., same inferences ✓
- Embedding "fallacies" in a logical context ✓
- Future work: Alignment of cognitive and logical theories
- Future work: system 1 and system 2 – prediction of intermediate steps

Talk Overview

Section 5

From nonmonotonic reasoning to belief revision

NMR and BR

Basically, belief revision deals with the problem of revising a (set or state) of beliefs K by new information A by applying a change operator $*$, obtaining a new (set or state) of beliefs K':

$$K' = K * A$$

Belief revision is nonmonotonic:

- We can have $K_1 \subseteq K_2$ but $K_1 * A \not\subseteq K_2 * A$;
- We can have $Cn(A) \subseteq Cn(B)$ but $K * A \not\subseteq K * B$.

Linking BR and NMR (and conditionals) via the Ramsey test

$$B \in K * A \quad \text{iff} \quad A \mid\!\sim_{(K)} B \quad \text{iff} \quad \Psi_K \models (B|A)$$

The core ideas of AGM theory

The AGM postulates are recommendations for rational belief change:

- The beliefs of the agent should be deductively closed, i.e., the agent should apply logical reasoning whenever possible.
- The change operation should be successfull. (This does not mean that the agent should believe everything!)
- In case of consistency, belief change should be performed via expansion.
- The result of belief change should only depend upon the semantical content of the new information.
- and more ...

Rankings – also a semantics for (iterated) belief revision

Theorem

A revision operator $*$ satisfies the basic axioms of AGM belief revision iff there is a total preorder $\leqslant_K$ (based on K) on the set of possible worlds such that

$$\mathit{Mod}(K * A) = \min(\mathit{Mod}(A), \leqslant_K),$$

i.e.,

$$K * A = \mathit{Th}(\min(\mathit{Mod}(A), \leqslant_K))$$

Ranking functions κ can also be conveniently used to implement such total preoders $\leqslant_K$ with $\mathit{Bel}(\kappa) = K$.

Problems with AGM

- Narrow logical framework: Classical propositional logic, no room for uncertainty
  $\rightarrow$ Richer epistemic frameworks?
- One-step revision: AGM belief revision does not consider changes of epistemic states (i.e., total preorders) nor revision strategies
  $\rightarrow$ Iterated revision
- New information: Only one proposition – what about sets of propositions, conditional statements, sets of conditionals?
  $\rightarrow$ Conditional and multiple belief revision

Advanced belief revision for ranking functions

Belief revision task for OCF

Given a prior OCF κ and some new information consisting of a set of conditionals $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, find a

posterior OCF $\kappa^* = \kappa * \Delta$

such that $\kappa^* \models \Delta$ and the revision complies with the core ideas of AGM.

This task involves

- iterated revision, since an epistemic state κ is changed;
- conditional revision, since the prior is revised by conditional information;
- multiple revision, since Δ can be a set of plausible propositions by setting $A \equiv (A|\top)$.

A principle of conditional preservation for ranking functions

OCF principle of conditional preservation (OCF-PCP)

Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ and $\Omega' = \{\omega'_1, \ldots, \omega'_m\}$ be two sets of possible worlds (not necessarily different).
If for each conditional $(B_i|A_i)$ in Δ, Ω and Ω' behave the same, i.e., they show the same number of verifications resp. falsifications, then prior κ and posterior κ^* are balanced by

$$\begin{aligned}(\kappa(\omega_1) + \ldots + \kappa(\omega_m)) - (\kappa(\omega'_1) + \ldots + \kappa(\omega'_m))\\ = (\kappa^*(\omega_1) + \ldots + \kappa^*(\omega_m)) - (\kappa^*(\omega'_1) + \ldots + \kappa^*(\omega'_m))\end{aligned}$$

[Kern-Isberner 2001]

If Ω and Ω' behave the same with respect to Δ, then their differences are the same in prior and posterior OCF.

A simple principle of conditional preservation

The general principle of conditional preservation yields a simple, straightforward consequence:

Simple PCP

(SCondPres) If two possible worlds $\omega_1, \omega_2 \in \Omega$ verify resp. falsify exactly the same conditionals in Δ, then $\kappa^*(\omega_1) - \kappa(\omega_1) = \kappa^*(\omega_2) - \kappa(\omega_2)$.

(SCondPres) claims that the amount of change between prior and posterior epistemic state depends only on the conditionals in the new information set, more precisely, on the so-called conditional structure of the respective world.

C-revisions

...are revisions that satisfy the principle of conditional preservation.

New information $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$

OCF c-revision

$$\kappa^* = \kappa * \Delta \ : \ \kappa^*(\omega) = \kappa_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^-,$$

κ_i^-'s have to be chosen appropriately to ensure $\kappa^* \models \mathcal{R}$ (Success).

κ_i^- is the impact that conditional $(B_i|A_i)$ has in the change process.
(Success) is satisfied iff for all $i, 1 \leqslant i \leqslant n$,

$$\kappa_i^- > \min_{\omega \models A_i B_i} (\kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^-) - \min_{\omega \models A_i \overline{B}_i} (\kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^-).$$

Revision techniques for inductive reasoning

Nonmonotonic inductive reasoning based on belief revision techniques is possible by taking a uniform epistemic state $\Psi_u (= \kappa_u)$[4] as prior epistemic state:

Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a finite set of conditionals;

$$\kappa_u * \Delta$$

allows model-based inductive inference.

C-representations of $\Delta \equiv$ c-revisions of κ_u by Δ

- improving system Z [Pearl 1990]
- generalizing system Z^* [Goldszmidt, Morris & Pearl 1993])

This allows for a seamless integration of reasoning and revision.

[4] $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$

Extended example – birds' scenario

f - *flying*	b - *birds*
p - *penguins*	w - *winged animal*
k - *kiwis*	d - *doves*.

Δ

r_1	:	$(f\|b)$	*birds fly*
r_2	:	$(b\|p)$	*penguins are birds*
r_3	:	$(\overline{f}\|p)$	*penguins do not fly*
r_4	:	$(w\|b)$	*birds have wings*
r_5	:	$(b\|k)$	*kiwis are birds*
r_6	:	$(b\|d)$	*doves are birds*

Strict knowledge: *Penguins, kiwis, and doves are pairwise exclusive*.

Birds' scenario (cont'd)

Let $*$ be a c-revision for OCF.
Initial epistemic state:

$$\kappa = \kappa_u * \Delta,$$

where κ_u is the uniform ranking function.
Here we have:

Initial state obtained via $\kappa_u * \Delta$

$$\kappa_u * \Delta \models (\overline{f}|p), (w|p), (w|d), (w|k), (f|d), (f|k)$$

Penguin-birds do not fly, but all birds – penguins, kiwis, and doves – inherit the property of having wings from their superclass *birds*; kiwis and doves are supposed to fly.

(Note that exactly the same beliefs hold for kiwis and doves!)

Birds' scenario (cont'd)

Current epistemic state: $\kappa = \kappa_u * \Delta$

- Revision: Now, the agent gets to know that having wings is false for kiwis - kiwis do not possess wings:

  $$\kappa_1^* = \kappa_u * (\Delta \cup \{(\overline{w}|k)\})$$

  $\kappa_1^* \models (b|k), (f|k)$ – kiwis are birds, kiwis fly.

- Update: The agent learns from the news, that, due to some mysterious illness that has occurred recently among doves, the wings of newborn doves are nearly completely mutilated:

  $$\kappa_2^* = \kappa * \{(\overline{w}|d)\} = (\kappa_u * \Delta) * \{(\overline{w}|d)\}$$

  $\kappa_2^* \not\models (b|d), (f|d)$ – now it is unknown whether doves are birds or fly

Talk Overview

Section 6

Probabilistic belief revision

200 years before AGM ...

Considering the task of belief change is not new: About 200 years before AGM theory, Bayes came up with his famous rule in probabilistics:

$$P(B|A) = \frac{P(A \wedge B)}{P(A)}.$$

Actually, Bayesian conditioning fulfills the core ideas of AGM theory, but obviously, the contexts of the theories (changing a code of law for AGM vs. random experiments and chances – e.g., in gambling – for Bayes) seemed to be too diverse to realise a strong connection.

The general task of belief change

However, from a formal resp. epistemic point of view, the tasks are similar if not identical:

General task of belief change

Given some (prior) epistemic state Ψ and some new information I, change beliefs rationally by applying a change operator $*$ to obtain a (posterior) epistemic state Ψ':

$$\Psi * I = \Psi'$$

AGM	: $\Psi = K$	set of propositional beliefs
Bayes	: $\Psi = P$	probability distribution
both	: $I = A$	propositional belief

An advanced probabilistic belief change task

The agent wants to adapt her probabilistic belief state P to a set of new conditional beliefs $\mathcal{R} = \{(B_1|A_1)[x_1], \ldots, (B_1|A_1)[x_1]\}$ – what is $P * \{(B_1|A_1)[x_1], \ldots, (B_1|A_1)[x_1]\}$?[5]

Use cross-entropy = information distance (= Kullback-Leibler-divergence)

$$R(Q, P) = \sum_{\omega \in \Omega} Q(\omega) \log \frac{Q(\omega)}{P(\omega)}$$

ME belief change

Given some prior P and some new $\mathcal{R}$, choose the unique distribution

$$P^* = P *_{ME} \mathcal{R} = \arg\min_{Q \models \mathcal{R}} R(Q, P)$$

that satisfies $\mathcal{R}$ and has minimal information distance to P.

The principle of minimum cross-entropy (MinREnt) generalizes the principle of maximum entropy (MaxEnt).

[5] $\mathcal{R}$ may contain probabilistic conditionals as well as probabilistic and logical facts.

The big conditional picture

Probabilities		Ranking functions
Principle MaxEnt	$\longleftrightarrow$	c-representations
$\updownarrow$		$\updownarrow$
Principle MinREnt	$\longleftrightarrow$	c-revisions

Principles of conditional preservation

underlie all these reasoning mechanisms – they emerged from a probabilistic principle of conditional preservation that is one of the main guidelines for the principles of optimum entropy.

Back to commonsense reasoning . . .

Jeff Paris:
Common sense and maximum entropy.
Synthese 117, 75-93, 1999, Kluwer Academic Publishers

Theorem

Each (model-based) probabilistic inference process N that satisfies 7 principles of commonsense reasoning coincides with MaxEnt inference.

→ MaxEnt reasoning

- satisfies commonsense principles of probabilistic reasoning, and
- is the only probabilistic inference process doing so.

Principle of Maximum Entropy [Paris 2006]

- A *knowledge base* $\mathcal{R} = \{(B_1|A_1)[p_1], \ldots, (B_n|A_n)[p_n]\}$ is a set of conditional statements of the form:

  "If A holds, then B follows with probability p."

- A probability distribution $\mathcal{P}$ can be seen as a formalization of the *belief state* of a reasoner with knowledge $\mathcal{R}$ iff

$$\mathcal{P}(B_i|A_i) = p_i \qquad \forall\, i = 1, \ldots, n.$$

Definition

The *maximum entropy distribution* $\mathcal{P}^{\mathsf{ME}}_{\mathcal{R}}$ is the unique probability distribution

$$\mathcal{P}^{\mathsf{ME}}_{\mathcal{R}} = \arg\max_{\mathcal{P} \models \mathcal{R}} - \sum_{\omega \in \Omega} \mathcal{P}(\omega) \cdot \log_2(\mathcal{P}(\omega))$$

that satisfies $\mathcal{R}$ and adds as few information as possible.

Translation of Syllogisms to Probabilistic Conditionals

(joint work with Marco Wilhelm)

Syllogism	**Conditional**
All A's are B's	$(B\|A)[1]$
No A's are B's	$(B\|A)[0]$
Some A's are B's	$(B\|A)[0.65]$
Some A's are *not* B's	$(B\|A)[0.15]$

How Does the MaxEnt Model Work?

Let $\mathcal{R}$ be a set of conditionals derived from given syllogisms.

1. Calculate the maximum entropy distribution $\mathcal{P}^{\mathsf{ME}}_{\mathcal{R}}$.
2. For every query

   $\{$**All** $|$ **No** $|$ **Some** $|$ **Some not**$\}$ **A's are B's?**

   calculate $p = \mathcal{P}^{\mathsf{ME}}_{\mathcal{R}}(B|A)$.
3. If the quantifier is $\left\{ \begin{array}{c} \textbf{All} \\ \textbf{No} \\ \textbf{Some} \\ \textbf{Some Not} \end{array} \right\}$ and $\left\{ \begin{array}{c} \mathbf{p = 1} \\ \mathbf{p = 0} \\ \mathbf{p \in [0.65 \pm t]} \\ \mathbf{p \in [0.15 \pm t]} \end{array} \right\}$,

   then *accept* the answer. (here: threshold $t = 0.1$)
4. If no answer is accepted, return NVC. Otherwise, return any accepted answer.

Performance of MaxEnt at the CogSci 2019 Challenge

- The MaxEnt model for syllogisms was evaluated on benchmark examples and proved to be comparable to best models.
- MaxEnt performs particularly well when no training data is available.
- When enough training data is available, a MaxEnt-MFA hybrid model performed best; prediction accuracy of MaxEnt MFA Hybrid: 44.03 %.

Talk Overview

Section 7

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